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Stochastic methods in dispersion theory

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ABSTRACT

In this study we show how methods from the theory of stochastic processes can be applied to problems in dispersion theory.
First, we show that Taylor dispersion with adsorbing boundaries is easily transformed into a new Taylor dispersion problem without adsorbing boundaries. The transformed problem can then be solved using any of the traditional methods used for Taylor dispersion.
Secondly, we consider the dispersion of particles in a channel (between parallel plates) with one partially adsorbing surface and one perfectly reflecting boundary. We determine the exact law of the position of adsorption for an arbitrary channel flow in terms of an infinite series of iterated integrals of the flow field, which is assumed to be a function of the cross-channel coordinate only. We also consider the case of shear flow over an adsorbing plane, by taking the limit where one of the boundaries is taken to infinity

1. Introduction

Most problems in dispersion theory can be studied most efficiently using the theory of stochastic processes. In this study, we consider two problems from this point of view to illustrate the approach.

In Section 2 we show the equivalence of Taylor dispersion with and without adsorbing boundaries, and in Section 3 we shear dispersion over a partially adsorbing surface, but without assuming the Taylor dispersion limit.

2. The equivalence of Taylor dispersion with and without adsorbing boundaries

The standard Taylor dispersion problem [1–3] is to determine the approximate long-time distribution of some passive tracer with density ϕ in a channel or pipe $\Omega = \mathbb{R} \times S$, where S is some connected smooth (usually one- or two-dimensional) Euclidean manifold. The governing equation in PDE form is

$$\frac{\partial \phi}{\partial t} + \bar{\nabla} \cdot (V\phi - \bar{D} \cdot \bar{\nabla}\phi) = 0, \tag{1}$$

where $\bar{\nabla}$ acts in Ω, t is time, $V(y)$, $y \in S$, is the velocity field, and $\bar{D}(y)$, $y \in S$, is the diffusivity tensor, usually with zero-flux boundary conditions on the walls. Note that, in the standard problem, V does not depend on time, or on the displacement x along the pipe, though in some cases this is easily relaxed [4], for example for a time-periodic velocity field [5].

In (1), it is usual to assume that $\bar{D}(y)$ has the following form:

$$\bar{D}(y) = \begin{pmatrix} D_x(y) & 0 \\ 0 & D(y) \end{pmatrix}, \tag{2}$$

where $D_x(y)$ is the diffusivity along the pipe, and $D(y)$ is the diffusivity tensor in S. This form ensures that the particle motion along the pipe is a 'slave' to the particle motion in S. It is also convenient to write

$$V(y) = \big(u(y), v(y)\big), \qquad \bar{\nabla} = \left(\frac{\partial}{\partial x}, \nabla\right), \tag{3}$$

where in each case the first component is the x or pipe part, and the second component is the S part.

For a good review of some of the standard methods in dispersion theory, see [6], who consider applications to dispersion in rivers and estuaries.

When the walls are adsorbing, or partially adsorbing, so that there is a non-zero flux of particles into the boundary that never return to the flow, the Taylor dispersion problem, at first sight, appears to be more complex than that with non-adsorbing boundaries [7–10]. There are applications where S is not simply connected, for example see [11] who consider the effects of wall absorption on dispersion in annular flows. One application of this work is to dispersion of solutes in blood flow in a catheterised artery.

The purpose of this note is to show that the Taylor dispersion problem with adsorbing boundaries is, in fact, equivalent to a variant problem with non-adsorbing boundaries, which can be studied using all the usual techniques for that case. The basic idea is simple, and well-known in probability theory [12]: just follow the particles that are not adsorbed up to a time T, say. The *conditioned* diffusion equation for the particles that survive up to time T

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is very similar to the original equation with a modified drift term and non-adsorbing boundary conditions.

2.1. Conditioned on survival

Consider (1) for $t \in [0, T]$, but with a partially adsorbing boundary, namely,

$$\bar{n} \cdot (V\phi - \bar{D} \cdot \bar{\nabla}\phi) = \rho\phi, \tag{4}$$

where $\bar{n}$ is a unit outward normal on $\partial\Omega$, and $\rho \geqslant 0$ is the adsorption rate, which may depend on $y \in \partial S$. Suppose that we are interested only in the motion of the particles that survive at least to time T. To find the conditioned diffusion equation for these particles, we first need to find the probability $p(y, t, T)$ that a particle starting at $y \in S$ at time $t \leqslant T$ survives until time T.

Recently, Taylor dispersion with adsorbing boundaries has been revisited [9] using methods familiar in physics. The goal of this work was to find a much more direct method of determining the moments of the particle displacement along the pipe than that used in earlier work [7]. In particular, the new method [9] makes it very much easier to find higher-order moments, e.g. the skewness, which requires the third moment of X_t. The new method [9], however, misses the point that we can condition on survival, and convert the adsorbing boundary problem to the standard one.

2.1.1. The survival probability

In [9], the authors assume constant scalar D and assume that the only advection is along the pipe; however, here, we consider the general case.

Consider (1) with an adsorbing boundary condition (4), and recall that the motion along the pipe is driven by the motion of the particles projected on S, but there is no feedback to S. So we can consider the motion in S in isolation. This is governed by

$$\frac{\partial\psi}{\partial t} + \nabla \cdot (v\psi - D \cdot \nabla\psi) = 0, \tag{5}$$

where $\psi(y) \equiv \int_{\mathbb{R}} \phi(x, y)\, dx$ is the particle density in S, namely the projection of ϕ on S. The boundary condition (4), due to the symmetry of Ω, becomes

$$n \cdot (v\psi - D \cdot \nabla\psi) = \rho\psi, \tag{6}$$

where n is a unit outward normal on ∂S.

The solution to (5) with boundary condition (6) can be written as an eigenfunction expansion:

$$\psi(y, t) = \sum_{k=0}^{\infty} a_n \psi_k(y) \exp(-\lambda_k t) \tag{7}$$

where the a_k are constants depending on the initial conditions, ψ_k are orthogonal eigenfunctions in S, and λ_k are the eigenvalues. We assume that $a_0 \neq 0$ and $0 \leqslant \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \cdots$.

From (7), we find the propagator (as in [9]), namely

$$G(y', t, y) = \sum_{k=0}^{\infty} \psi_k(y')\psi_k(y) \exp(-\lambda_k t), \tag{8}$$

where for all i and j

$$\int_S \psi_i(y)\psi_j(y)\, dy = \delta_{ij}. \tag{9}$$

Thus the probability for a particle starting at y at time $t \leqslant T$ to survive until at least T, is

$$p(y, t, T) = \int_S G(y', T - t, y)\, dy. \tag{10}$$

In the Taylor dispersion limit, $T \gg \tau$, where τ is the relaxation time for diffusion in S. Rearranging (7), we find

$$\psi(y, t) = \exp(-\lambda_0 t)\left(a_0\psi_0(y) + \sum_{k=1}^{\infty} a_n \psi_k(y) \exp\big(-(\lambda_k - \lambda_0)t\big)\right). \tag{11}$$

Thus the first term in (11) dominates all the others provided that $t \gg \tau = (\lambda_1 - \lambda_0)^{-1}$. So, in this limit, we may approximate the survival probability by

$$p(y, t, T) = \psi_0(y)\left(\int_S \psi(y')\, dy'\right)\exp\big(-\lambda_k(T - t)\big) \times \big(1 + \mathrm{O}\big(\exp\big(-(T - t)/\tau\big)\big)\big). \tag{12}$$

So this approximation is valid up to within a few τ of the final time T.

2.1.2. Conditioning on survival

Now we have the survival probability, we can use the Doob h-transform to condition on survival. For applied mathematical discussions of this technique, see [4,13], or, for a much more thorough account, see [12].

For the current system, the modified diffusion equation in S for the particles conditioned to survive is

$$\frac{\partial\psi}{\partial t} + \nabla \cdot \big(\big(v + 2D \cdot \nabla(\log p)\big)\psi - D \cdot \nabla\psi\big) = 0, \tag{13}$$

with

$$n \cdot \big(\big(v + 2D \cdot \nabla(\log p)\big)\psi - D \cdot \nabla\psi\big) = 0. \tag{14}$$

That is, the conditioned diffusion has an additional drift that pushes particles away from the adsorbing boundaries. This is essentially an application of Bayes' theorem.

If we use the exact form of the additional drift, it is time dependent, but, apart from when $T - t = \mathrm{O}(\tau)$, we can use the approximate form, which leads to the following approximation

$$2D \cdot \nabla(\log p) = 2D \cdot \nabla(\log \psi_0)\big(1 + \mathrm{O}\big(\exp\big(-(T - t)/\tau\big)\big)\big). \tag{15}$$

In the Taylor limit, this approximate form can always be used.

The advection and diffusion along the pipe is unchanged, which completes the conditioned set of equations. The conditioned diffusion leads to a standard Taylor dispersion problem with non-adsorbing boundaries, and can be solved by any of the standard techniques.

2.1.3. The self-adjoint case

When $v = 0$, the diffusion equation (5) is self adjoint. In this case we can immediately see that the steady solution, in the Taylor limit, to the conditioned diffusion equation (13) with boundary condition (14) is

$$\psi \propto \psi_0^2, \tag{16}$$

as observed in [9].

2.2. Example: a simple channel flow

Take $S = [0, 1]$, and suppose that the particle is adsorbed on first contact with the boundaries, i.e. $\rho = \infty$. Also take $v = 0$ and $D = \frac{1}{2}$. In this case,

$$\psi_0(y) = \sqrt{2}\sin(\pi y). \tag{17}$$

Thus in the Taylor limit, the additional drift is

$$2D \cdot \nabla(\log p) = \frac{\partial}{\partial y}\log\psi_0(y) = \pi\cot(\pi y). \tag{18}$$

So the diffusion equation for the particles conditioned to survive is, in the Taylor limit,

$$\frac{\partial\psi}{\partial t}+\frac{\partial}{\partial y}\left(\pi\cot(\pi y)\psi-\frac{1}{2}\frac{\partial\psi}{\partial y}\right)=0, \tag{19}$$

with

$$\pi\cot(\pi y)\psi-\frac{1}{2}\frac{\partial\psi}{\partial y}=0 \tag{20}$$

on the boundaries.

The steady solution of (19) is

$$\psi\propto\psi_0^2(y)\propto\sin^2(\pi y), \tag{21}$$

from (16), as this system is self adjoint. So, for example, the mean particle advection velocity $\langle u\rangle$ along the pipe is given by

$$\langle u\rangle=\int_0^1 2\sin^2(\pi y)u(y)\,dy, \tag{22}$$

where $u(y)$, $y\in[0,1]$, is the advection velocity along the pipe.

In the case considered in [9], $u(y)=u_0y(1-y)$ after a trivial linear transformation. From (22), we find

$$\langle u\rangle=U\left(1+\frac{3}{\pi^2}\right), \tag{23}$$

where $U=u_0/6$ is the value of the mean drift for the case without adsorbing walls. So this shows the usual enhanced drift, and agrees with their result.

Clearly, any other aspect of Taylor dispersion can be determined in the usual way for systems with *non-adsorbing* boundaries [4], but we shall not consider this any further here.

3. Shear dispersion over a partially adsorbing surface

We consider a particle in a channel flow in which one boundary is partially adsorbing and the other is perfectly reflecting. We also allow the advection to have a cross-channel component, which could be, say, a drift term due to gravity in an atmospheric application. In such applications, one of the boundaries is taken to be at infinity, leading to the corresponding problem in a half-space.

In this section, we do not assume the Taylor-dispersion limit, so we cannot use the result from Section 2.

In PDE terms, this leads to a flux boundary condition. However, the current study will use Itô calculus, rather than the traditional method introduced by Taylor [1] and improved by Aris [3]. Thus, using stochastic calculus makes it possible to make direct use of the Ray–Knight theorem, which is at the heart of all Taylor dispersion problems, and significantly simplifies dispersion calculations [4].

In this note, we essentially unpack the Ray–Knight theorem in the context of dispersion theory, and demonstrate the advantage of using this theorem as a starting point for such calculations.

In Section 3.2, we consider both non-constant diffusivity and sedimentation, which lead to minor changes to the theory of Section 3.1.

Since we assume that the flow field is restricted to two dimensions, with one of these being the cross-channel direction, we restrict attention to two-dimensions, in a channel of width b, and assume that the flow field is a function only of the cross-channel coordinate (but is not necessarily unidirectional). The particle motion in the remaining dimension, which has no advection, is independent of the other components. Thus the exact form of this component is already known, and can be added at the end of the calculation, if required.

In many applications, the cross-channel component may not be a real fluid flow, but rather advection due to some applied field, which might be gravitational in large scale applications, like atmospheric flows, or electro-magnetic in microscopic applications. However, since the mathematical treatment is exactly the same in all of these cases, we shall refer to the cross-channel advection as a 'flow' in the analysis.

To clarify the problem, we write the PDE version of a cross-channel diffusion process for a particle confined to $y\in[0,b]$, namely

$$\frac{\partial\phi}{\partial t}=\frac{\partial}{\partial y}\left(D\frac{\partial\phi}{\partial y}\right), \tag{24}$$

where ϕ is the particle's probability density function with

$$\frac{\partial\phi}{\partial y}=k\phi\quad\text{on } y=0,\qquad\frac{\partial\phi}{\partial y}=0\quad\text{on } y=b, \tag{25}$$

k is the adsorption rate and D is the diffusivity. The corresponding problem for an annular cross-section pipe was considered by Mondal and Mazumder [14], who worked entirely with the PDE form. However, they restricted attention to the first few moments of the concentration distribution.

An example of a physical situation of this form is a layer of fluid flowing over a solid surface. On the free surface, the particle would normally have a reflecting boundary condition. So we are interested in the case where the particle can react with the lower solid surface, and in the final position of the adsorbed particle.

Since the main result is expressed directly in terms of integrals of the advection velocity, we may apply the result to many applications involving non-Newtonian fluids, including the flows of paints, lacquer, etc., or in mud flows.

In a channel with two partially adsorbing boundaries, the same analysis can be used if the flow field is symmetrical about the centre-line, in which case we can make use of the symmetry of the system and replace the centre-line by a reflecting boundary. We mention this briefly in Section 3.3

3.1. General shear dispersion

Consider the dispersion of a particle in a region $R=\{(x,y),\ x\in\mathbb{R},\ y\in[0,b]\}$ with a flow field $(u(y),0)$. To begin with, we assume that the vertical motion, before adsorption, is a standard Brownian motion with reflection at $y=0$ and $y=b$, and relax this assumption later, i.e. in this section the flow field is unidirectional. So the Itô stochastic differential equation for this motion is

$$dY_t=dB_t+dL_t(0)-dL_t(b), \tag{26}$$

where B is a standard Brownian motion (i.e. we have chosen units so that the diffusivity is $\frac{1}{2}$), L_t is the local time (which is the density of occupation time), and the $dL_t(0)$ and $dL_t(b)$ terms keep the particle in the channel.

Thus, before adsorption, the streamwise displacement X is given by

$$X_t=\int_0^t u(Y_s)\,ds+A_t, \tag{27}$$

where A is an independent Brownian motion representing the streamwise diffusion.

3.1.1. Particle released at $y=b$ above a partially adsorbing boundary at $y=0$

Suppose that the particle is released at $y=b$ at time $t=0$, and that the boundary condition at $y=b$ is perfect reflection and at $y=0$ the particle is adsorbed at a constant local-time rate $k>0$.

The x-position at the time T at which the particle is adsorbed at $y = 0$ is given by

$$X_T = \int_0^b u(y) L_T(y)\, dy + A_T. \tag{28}$$

In shear dispersion applications, the effect of the streamwise diffusion is negligible compared with the effect of the shear dispersion, and we shall neglect it from this point on.

The distribution of local time on the adsorbing boundary is EXP(k), and the Ray–Knight theorem implies that

$$L_T(y) \stackrel{d}{=} \big(W^{(1)}\big(y + (2k)^{-1}\big)\big)^2 + \big(W^{(2)}\big(y + (2k)^{-1}\big)\big)^2, \tag{29}$$

where $W^{(1)}$ and $W^{(2)}$ are independent standard Brownian motions, $W^{(1)}(0) = W^{(2)}(0) = 0$, and '$\stackrel{d}{=}$' means 'equal in law'.

We wish to find the law of X_T, or, more precisely, evaluate

$$\Phi^2 \equiv E\left[\exp\left(-\frac{1}{2}\alpha^2 X_T\right) \,\middle|\, (X_0, Y_0) = (0, b)\right]. \tag{30}$$

Since, from (29), we see that X_T can be split into two independent and identically distributed parts, we only determine

$$\Phi = E\left[\exp\left(-\frac{1}{2}\alpha^2 \int_0^b \big(W^{(1)}\big(y + (2k)^{-1}\big)\big)^2\, dy\right)\right]. \tag{31}$$

We can determine Φ exactly, since, to a physicist, this is a quadratic field theory, as we have squares of Gaussian processes. Using Wick's theorem, the diagramatic expansion of Φ is a series of loop diagrams. It is better, however, to use the exponential formula, and work with $\log\Phi$, as then we have only to consider connected diagrams, in this case single loops, at each order. Following this procedure, we find

$$E\left[\exp\left(-\frac{1}{2}\alpha^2 X_T\right) \,\middle|\, (X_0, Y_0) = (0, b)\right] = \exp\left(\sum_{n=1}^{\infty} (-\alpha^2)^n K_n\right), \tag{32}$$

where

$$K_n = \frac{1}{n}\int_0^b \cdots \int_0^b \left(\prod_{i=1}^{n} \big(\min(y_i, y_{i+1}) + (2k)^{-1}\big)\right)\left(\prod_{i=1}^{n} u(y_i)\, dy_i\right) \tag{33}$$

and $y_{n+1} = y_1$. The $\frac{1}{2}K_n$ are the averages of the loop diagrams of $\log\Phi$, and the 'min' come from

$$E\big[W^{(1)}\big(y_i + (2k)^{-1}\big) W^{(1)}\big(y_{i+1} + (2k)^{-1}\big)\big] = \min(y_i, y_{i+1}) + (2k)^{-1}. \tag{34}$$

Observe that we can split the streamwise displacement into two independent parts, namely

$$X(H_0), \qquad X(T) - X(H_0), \tag{35}$$

where $t = H_0$ is the first time that the particle hits $y = 0$. From the Ray–Knight theorem, the local time for the first part is

$$L_T(y) \stackrel{d}{=} \big(W^{(1)}(y)\big)^2 + \big(W^{(2)}(y)\big)^2, \tag{36}$$

which is, as expected, also the $k \to \infty$ limit of (29). Thus

$$E\left[\exp\left(-\frac{1}{2}\alpha^2 X(H_0)\right) \,\middle|\, (X_0, Y_0) = (0, b)\right] = \exp\left(\sum_{n=1}^{\infty} (-\alpha^2)^n M_n\right), \tag{37}$$

where

$$M_n = \frac{1}{n}\int_0^b \cdots \int_0^b \left(\prod_{i=1}^{n} \min(y_i, y_{i+1})\right)\left(\prod_{i=1}^{n} u(y_i)\, dy_i\right). \tag{38}$$

3.1.2. Better forms of K_n for computer algebra manipulators

When evaluating the K_n in a special case using standard computer algebra manipulators, we found that many algebra manipulators could not cope with the general form of (33), as these systems could not handle integrals involving the function $\min(\cdot,\cdot)$, or could cope only with some very trivial cases of such integrals. One would hope that at some stage all algebra manipulators will be able to deal directly with (33).

However, it is easy to transform the integrals into a form that can be handled. To do this we transform the integral for K_n into an integral over $0 \leqslant y_1 \leqslant y_2 \leqslant \cdots \leqslant y_n \leqslant b$ and sum the integrand over all permutations of the y_i. In the new form $\min(y_i, y_j) = y_{\min(i,j)}$. Thus each step is now in a form that can be easily performed by any computer algebra package.

Define

$$k[i_1, i_2, \ldots, i_n] = \int_0^b \int_0^{y_n} \cdots \int_0^{y_2} \left(\prod_{j=1}^{n} \big(y_{i_j} + (2k)^{-1}\big)\right)\left(\prod_{i=1}^{n} u(y_i) dy_i\right). \tag{39}$$

We now give the first few K_n in terms of the function k to highlight the structure:

$$K_1 = k[1], \tag{40}$$

$$K_2 = k[1, 1], \tag{41}$$

$$K_3 = 2k[1, 1, 2], \tag{42}$$

$$K_4 = 2k[1, 1, 2, 2] + 4k[1, 1, 2, 3], \tag{43}$$

$$K_5 = 8k[1, 1, 2, 2, 3] + 4k[1, 1, 2, 2, 4] + 4k[1, 1, 2, 3, 3] + 8k[1, 1, 2, 3, 4], \tag{44}$$

$$\begin{aligned} K_6 = {} & 12k[1, 1, 2, 2, 3, 3] + 32k[1, 1, 2, 2, 3, 4] + 16k[1, 1, 2, 2, 3, 5] \\ & + 4k[1, 1, 2, 2, 4, 4] + 8k[1, 1, 2, 2, 4, 5] + 16k[1, 1, 2, 3, 3, 4] \\ & + 8k[1, 1, 2, 3, 3, 5] + 8k[1, 1, 2, 3, 4, 4] \\ & + 16k[1, 1, 2, 3, 4, 5], \end{aligned} \tag{45}$$

$$\begin{aligned} K_7 = {} & 72k[1, 1, 2, 2, 3, 3, 4] + 48k[1, 1, 2, 2, 3, 3, 5] \\ & + 24k[1, 1, 2, 2, 3, 3, 6] + 48k[1, 1, 2, 2, 3, 4, 4] \\ & + 128k[1, 1, 2, 2, 3, 4, 5] + 64k[1, 1, 2, 2, 3, 4, 6] \\ & + 16k[1, 1, 2, 2, 3, 5, 5] + 32k[1, 1, 2, 2, 3, 5, 6] \\ & + 16k[1, 1, 2, 2, 4, 4, 5] + 8k[1, 1, 2, 2, 4, 4, 6] \\ & + 8k[1, 1, 2, 2, 4, 5, 5] + 16k[1, 1, 2, 2, 4, 5, 6] \\ & + 24k[1, 1, 2, 3, 3, 4, 4] + 64k[1, 1, 2, 3, 3, 4, 5] \\ & + 32k[1, 1, 2, 3, 3, 4, 6] + 8k[1, 1, 2, 3, 3, 5, 5] \\ & + 16k[1, 1, 2, 3, 3, 5, 6] + 32k[1, 1, 2, 3, 4, 4, 5] \\ & + 16k[1, 1, 2, 3, 4, 4, 6] + 16k[1, 1, 2, 3, 4, 5, 5] \\ & + 32k[1, 1, 2, 3, 4, 5, 6]. \end{aligned} \tag{46}$$

These expressions for K_n rapidly become large as n increases and even the expression for K_8 has 51 terms. However, with modern computers it is possible to determine exactly many K_n for flow fields of practical interest.

The corresponding expressions for M_n follow from taking the $k \to \infty$ limits of the above.

3.1.3. Particle released at $y = 0$ on a partially adsorbing boundary at $y = 0$

Again, we suppose that the boundary at $y = b$ is perfectly reflecting and the boundary at $y = 0$ is partially adsorbing, but this time we release the particle at $y = 0$.

This is almost the same as the calculation of Section 3.1.1, but we lack the contribution from $0 \leqslant t \leqslant H_0$. That is, the distribution of X_T is identical to that of $X(T) - X(H_0)$ in Section 3.1.1. Thus

$$E\left[\exp\left(-\frac{1}{2}\alpha^2 X_T\right) \,\middle|\, (X_0, Y_0) = (0,0)\right] = \exp\left(\sum_{n=1}^{\infty}(-\alpha^2)^n J_n\right), \quad (47)$$

where $J_n = K_n - M_n$.

3.2. Variable diffusion coefficient and vertical drift

The above analysis can easily be extended to consider the corresponding system for which the diffusion coefficient and the vertical drift are both y-dependent functions. Furthermore, this can be achieved without having to rederive the results of Section 3.1, as the new governing equations can be exactly transformed into those already solved above, using the well known 'scale and speed' transformations often used in the theory of stochastic processes. Thus, this method is not restricted to unidirectional flows, so the effects of sedimentation and other cross-channel advection effects, are easily included.

The annular-pipe geometry studied by Mondal and Mazumder [14] can also be considered by a 'scale and speed' transformation of the results here, provided that the along-pipe flow is a function only of the radial coordinate. In this case, the system is essentially one-dimensional.

3.2.1. Example: sedimentation

Suppose that the vertical diffusivity D is constant, as before, but the velocity field is replaced by $(u(y), v(y))$. The vertical velocity v might model a sedimentation velocity, for example. Then, using the 'scale and speed' transformations, we find that the corresponding results to (32), (37) and (47) have $u(y)$ in the definitions of K_n, M_n and J_n replaced by

$$U(y) = u(y)\exp\left(D^{-1}\int_0^y v(y')\,dy'\right). \quad (48)$$

3.3. Symmetrical flows

In general, a channel with two partially adsorbing boundaries is mathematically more complex. One important exception, however, is the case of a symmetrical flow, as we can split the channel along its centre-line and use the theory of Section 3.1, since the boundary condition on the centre-line is perfect reflection.

4. Conclusions

In Section 2, we have shown the equivalence of Taylor dispersion with and without adsorbing boundaries. Thus, the standard methods of Taylor dispersion theory can be used for the surviving particles by considering the governing equation conditioned on survival up to some arbitrary time T.

Time periodic flows can also be treated in the same way, by adding a periodic dimension to S, with, say, constant advection and no diffusion. Thus the probabilistic technique used in [15] for a time-periodic flow can easily be extended to include adsorbing boundaries.

In Section 3, we have explored application of the Ray–Knight theorem to Taylor dispersion problems involving a partially adsorbing surface (i.e. one with a flux boundary condition). In this case, it is possible to find the exact law for the point on the adsorbing boundary where the particle finally sticks, in terms of infinite series using standard tricks from quadratic field theory.

Usually, in Taylor dispersion problems, only the first few moments are determined, but here we have found explicit expressions for all the moments of X_T. Using standard algebra manipulation systems, large numbers of moments could be exactly computed for any simple channel flow field.

In applications, X_T might be the position where some contaminant was deposited. This contaminant might just stick to the boundary or react with it. If the material is not permanently attached to the boundary, we can use the theory of Section 3.1 for each free phase of the particle motion, and mix with an arbitrary law for the time that the particle is attached to the boundary. This approach might be a natural way to model dispersion in systems with boundary retention effects, and lead to extensions of the work of Purnama [16].

It is likely that many other dispersion problems could be solved exactly in a similar form. Such exact results highlight the structure of dispersion processes, and show how they approach the Taylor dispersion limit, as well as being useful in interesting special cases.

The stochastic processes approach to these problems, once learned, is much easier to use than the PDE alternatives, and the ease of use often leads more naturally to more general results.

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